[2]

$$\text{①}\quad (2D-2)[x] + (3D+2)[y] = 0$$
$$\text{②}\quad D[x] + (D+1)[y] = 25t\cos t$$

(2)

$-(D+1)[①]$ $\quad -(D+1)(2D-2)[x] - (D+1)(3D+2)[y] = 0$

$(3D+2)[②]$ $\quad D(3D+2)[x] + (D+1)(3D+2)[y] = 75\cos t - 75t\sin t + 50t\cos t$

(3)

$$(-2D^2 + 2 + 3D^2 + 2D)[x] = (50t+75)\cos t - 75t\sin t$$

$$x'' + 2x' + 2x = (D^2 + 2D + 2)[x] = (50t+75)\cos t - 75t\sin t$$ (2)

$$r^2 + 2r + 2 = 0 \rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$x_h = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$ (2)

$$x_p = (At+B)\cos t + (Ct+D)\sin t$$ (1½)

$$x_p' = A\cos t + (-At-B)\sin t + (Ct+D)\cos t + C\sin t$$

$$= (Ct+A+D)\cos t + (-At-B+C)\sin t$$ (3)

$$x_p'' = C\cos t + (-Ct-A-D)\sin t + (-At-B+C)\cos t - A\sin t$$

(3)

$$x_p'' = (-At-B+2C)\cos t + (-Ct-2A-D)\sin t$$
$$+2x_p' \quad +(2Ct+2A+2D)\cos t + (-2At-2B+2C)\sin t$$
$$+2x_p \quad +(2At+2B)\cos t + (2Ct+2D)\sin t$$ (3)

$$= ((A+2C)t + (2A+B+2C+2D))\cos t + ((-2A+C)t + (-2A-2B+2C+D))\sin t$$ (3)

$$= (50t+75)\cos t - 75t\sin t$$

③ $A + 2C = 50$

④ $-2A + C = -75$ (1½)

$2*$③ $2A + 4C = 100$

$5C = 25$

$C = 5$

$A + 10 = 50 \rightarrow A = 40$

$80 + B + 10 + 2D = 75$

$-80 - 2B + 10 + D = 0$

⑤ $B + 2D = -15$

⑥ $-2B + D = 70$ (2)

$2*$⑤ $2B + 4D = -30$

$5D = 40 \rightarrow D = 8$

$B + 16 = -15$

$B = -31$

$$x = \underline{(40t-31)\cos t + (5t+8)\sin t} \; \underline{+ c_1 e^{-t}\cos t + c_2 e^{-t}\sin t}$$

8 $\quad 1\tfrac{1}{2}$

$$\boxed{\begin{array}{ll} D[①] & D(2D-2)[x] + D(3D+2)[y] = 0 \\ -(2D-2)[②] & -D(2D-2)[x] - (D+1)(2D-2)[y] = -50\cos t \\ & \qquad +50t\sin t \\ & \qquad +50t\cos t \end{array}}$$

3

$$\boxed{(D^2+2D+2)[y] = (50t-50)\cos t + 50t\sin t}$$

$1\tfrac{1}{2}$

$$y_h = k_1 e^{-t}\cos t + k_2 e^{-t}\sin t$$

$$y_p = (At+B)\cos t + (Ct+D)\sin t$$

⑦ $A + 2C = 50$	$-20 + B + 60 + 2D = -50$
⑧ $-2A + C = 50$ $\quad 1\tfrac{1}{2}$	$20 - 2B + 60 + D = 0$
2*⑦ $2A + 4C = 100$	⑨ $B + 2D = -90$
$5C = 150$	⑩ $-2B + D = -80$ $\quad 2$
$C = 30$	2*⑨ $2B + 4D = -180$
$A + 60 = 50 \rightarrow A = -10$	$5D = -260$
	$D = -52$
	$B - 104 = -90$
	$B = 14$

$$y = \underline{(-10t+14)\cos t + (30t-52)\sin t} + \underline{k_1 e^{-t}\cos t + k_2 e^{-t}\sin t}$$

8 $\quad 1\tfrac{1}{2}$

$$\begin{array}{lllll} x' = & 40\cos t & + (-40t+31)\sin t & - c_1 e^{-t}\cos t & - c_1 e^{-t}\sin t \\ & +(5t+8)\cos t & +5\sin t & + c_2 e^{-t}\cos t & - c_2 e^{-t}\sin t \quad ④ \\ +y' & -10\cos t & + (10t-14)\sin t & - k_1 e^{-t}\cos t & - k_1 e^{-t}\sin t \\ & +(30t-52)\cos t & +30\sin t & + k_2 e^{-t}\cos t & - k_2 e^{-t}\sin t \quad ④ \\ +y & +(-10t+14)\cos t & +(30t-52)\sin t & + k_1 e^{-t}\cos t & + k_2 e^{-t}\sin t \end{array}$$

$$\boxed{\begin{array}{ll} 25t\cos t & +(-c_1+c_2+k_2)e^{-t}\cos t \\ & +(-c_1-c_2-k_1)e^{-t}\sin t \end{array}}$$

$= 25t\cos t$

4

(2)

$$\boxed{\begin{aligned} -c_1+c_2+k_2&=0 \\ -c_1-c_2-k_1&=0 \end{aligned}}\begin{aligned} &\longrightarrow k_2=c_1-c_2 \\ &\longrightarrow k_1=-c_1-c_2 \end{aligned}$$

(3)

$$\boxed{\begin{aligned} x&=(40t-31)\cos t+(5t+8)\sin t+c_1e^{-t}\cos t+c_2e^{-t}\sin t \\ y&=(-10t+14)\cos t+(30t-52)\sin t+(-c_1-c_2)e^{-t}\cos t \\ &\quad+(c_1-c_2)e^{-t}\sin t \end{aligned}}$$

[3] [a] (3) $x(n(n-1)x^{n-2}) + (2+x\tan x)nx^{n-1} + (\tan x)x^n = 0$

$n(n-1)x^{n-1} + 2nx^{n-1} + nx^n\tan x + x^n\tan x = 0$

$n(n-1+2)x^{n-1} + (n+1)x^n\tan x = 0$

(3) $(n+1)(nx^{n-1} + x^n\tan x) = 0$

$n+1 = 0 \longrightarrow n = -1$ (1½)

[b] $y_2 = vx^{-1}$ (1½)

$y_2' = v'x^{-1} - vx^{-2}$ (1½)

$$y_2'' = v''x^{-1} - v'x^{-2} - v'x^{-2} + 2vx^{-3} = v''x^{-1} - v'(2x^{-2}) + v(2x^{-3})$$ (3)

$$xy_2'' + (2+x\tan x)y_2' + (\tan x)y$$

$$= \boxed{\begin{array}{lll} v'' - v'(2x^{-1}) & & +v(2x^{-2}) \\ & +v'(2x^{-1}+\tan x) & -v(2x^{-2}+x^{-1}\tan x) \\ & & +v(\quad x^{-1}\tan x) \end{array}}$$ (3)

$= v'' + v'\tan x$ (3) CHECKPOINT: NO V WITHOUT ' (1½)

$u = v' \longrightarrow \frac{du}{dx} + u\tan x = 0$

$\int \frac{1}{u}\,du = \int -\tan x\,dx$ (1½) CHECKPOINT: SEPARABLE (1½)

$\ln|u| = \ln|\cos x|$

$v' = u = \cos x$ (1½)

$v = \sin x$ (1½)

$y_1 = x^{-1}, \; y_2 = x^{-1}\sin x$ (1½)

$$W = \begin{vmatrix} x^{-1} & x^{-1}\sin x \\ -x^{-2} & -x^{-2}\sin x + x^{-1}\cos x \end{vmatrix} = x^{-2}\cos x$$ (3)

$$g = \frac{\sec^2 x}{x} = x^{-1}\sec^2 x$$ (1½)

$$y_p = -x^{-1}\int \frac{(x^{-1}\sec^2 x)(x^{-1}\sin x)}{x^{-2}\cos x}\,dx + x^{-1}\sin x\int \frac{(x^{-1}\sec^2 x)\,x^{-1}}{x^{-2}\cos x}\,dx$$

$$= -x^{-1}\underline{\int \sec^2 x\tan x\,dx} \; 3 + x^{-1}\sin x\,\underline{\int \sec^3 x\,dx} \; 3$$

$$\boxed{\begin{matrix} u = \tan x \\ \int u\,du \end{matrix}} \; 1\tfrac{1}{2}$$

$$= \boxed{-x^{-1}\underline{\left(\tfrac{1}{2}\tan^2 x\right)} \; 1\tfrac{1}{2} + x^{-1}\sin x\,\boxed{\left(\tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\ln|\sec x + \tan x|\right)} \; 1\tfrac{1}{2}} \; 3$$

$$= -\tfrac{1}{2}x^{-1}\tan^2 x + \tfrac{1}{2}x^{-1}\tan^2 x + \tfrac{1}{2}x^{-1}(\sin x)\ln|\sec x + \tan x|$$

$$= \tfrac{1}{2}x^{-1}(\sin x)\ln|\sec x + \tan x|$$

$$y = \underbrace{\boxed{\tfrac{1}{2}x^{-1}(\sin x)\ln|\sec x + \tan x|}}_{2} + \underbrace{\boxed{c_1 x^{-1} + c_2 x^{-1}\sin x}}_{1\frac{1}{2}}$$

[4] $4r^4-7r^2+2r+5=0$

$$
\begin{array}{r|rrrrr}
-1 & 4 & 0 & -7 & 2 & 5 \\
 & & -4 & 4 & 3 & -5 \\
\hline
-1 & 4 & -4 & -3 & 5 & 0 \\
 & & -4 & 8 & -5 & \\
\hline
 & 4 & -8 & 5 & 0 &
\end{array}
$$

2 2

$4r^2-8r+5=0$

$$r=\frac{8\pm\sqrt{64-80}}{8}=\frac{8\pm4i}{8}=1\pm\tfrac{1}{2}i$$

$$x_h=c_1e^{-t}+c_2te^{-t}+c_3e^{t}\cos\tfrac{1}{2}t+c_4e^{t}\sin\tfrac{1}{2}t$$

1 2

$$x_p=((At+B)e^{-t})t^2+C\cos\tfrac{1}{2}t+D\sin\tfrac{1}{2}t+Et^2+Ft+G$$

$$=(At^3+Bt^2)e^{-t}+C\cos\tfrac{1}{2}t+D\sin\tfrac{1}{2}t+Et^2+Ft+G$$

3 1½ 1½

$$x_p'=(-At^3-Bt^2+3At^2+2Bt)e^{-t}+\tfrac{1}{2}D\cos\tfrac{1}{2}t-\tfrac{1}{2}C\sin\tfrac{1}{2}t+2Et+F$$

$$=(-At^3+(3A-B)t^2+2Bt)e^{-t}+\tfrac{1}{2}D\cos\tfrac{1}{2}t-\tfrac{1}{2}C\sin\tfrac{1}{2}t+2Et+F$$

3 1½ 1

$$x_p''=(At^3+(-3A+B)t^2-2Bt-3At^2+(6A-2B)t+2B)e^{-t}-\tfrac{1}{4}C\cos\tfrac{1}{2}t-\tfrac{1}{4}D\sin\tfrac{1}{2}t+2E$$

$$=(At^3+(-6A+B)t^2+(6A-4B)t+2B)e^{-t}-\tfrac{1}{4}C\cos\tfrac{1}{2}t-\tfrac{1}{4}D\sin\tfrac{1}{2}t+2E$$

4 1½ ½

$$x_p'''=(-At^3+(6A-B)t^2+(-6A+4B)t-2B+3At^2+(-12A+2B)t+(6A-4B))e^{-t}-\tfrac{1}{8}D\cos\tfrac{1}{2}t+\tfrac{1}{8}C\sin\tfrac{1}{2}t$$

$$=(-At^3+(9A-B)t^2+(-18A+6B)t+(6A-6B))e^{-t}-\tfrac{1}{8}D\cos\tfrac{1}{2}t+\tfrac{1}{8}C\sin\tfrac{1}{2}t$$

4 1½

$$x_p^{(4)}=(At^3+(-9A+B)t^2+(18A-6B)t+(-6A+6B)-3At^2+(18A-2B)t+(-18A+6B))e^{-t}+\tfrac{1}{16}C\cos\tfrac{1}{2}t+\tfrac{1}{16}D\sin\tfrac{1}{2}t$$

$$= \left(At^3 + (-12A+B)t^2 + (36A-8B)t + (-24A+12B)\right)e^{-t}$$ (4)

$$+ \tfrac{1}{16}C\cos\tfrac{1}{2}t + \tfrac{1}{16}D\sin\tfrac{1}{2}t$$ (1½)

$$\begin{aligned}
4x_p^{(4)} &= \left(4At^3 + (-48A+4B)t^2 + (144A-32B)t + (-96A+48B)\right)e^{-t} \\
&\quad + \tfrac{1}{4}C\cos\tfrac{1}{2}t + \tfrac{1}{4}D\sin\tfrac{1}{2}t \\
-7x_p'' &\quad + \left(-7At^3 + (42A-7B)t^2 + (-42A+28B)t - 14B\right)e^{-t} \\
&\quad + \tfrac{7}{4}C\cos\tfrac{1}{2}t + \tfrac{7}{4}D\sin\tfrac{1}{2}t - 14E \\
+2x_p' &\quad + \left(-2At^3 + (6A-2B)t^2 + 4Bt\right)e^{-t} \\
&\quad + D\cos\tfrac{1}{2}t - C\sin\tfrac{1}{2}t + 4Et + 2F \\
+5x_p &\quad + \left(5At^3 + 5Bt^2\right)e^{-t} \\
&\quad + 5C\cos\tfrac{1}{2}t + 5D\sin\tfrac{1}{2}t + 5Et^2 + 5Ft + 5G
\end{aligned}$$

(6)

$$= \left(102At + (-96A+34B)\right)e^{-t}$$ (4)

$$+ (7C+D)\cos\tfrac{1}{2}t + (-C+7D)\sin\tfrac{1}{2}t$$ (2)

$$+ 5Et^2 + (4E+5F)t + (-14E+2F+5G)$$ (2)

$$= 51te^{-t} + 50\sin\tfrac{1}{2}t - 25t^2$$

$102A = 51 \rightarrow A = \frac{1}{2}$

$-48 + 34B = 0 \rightarrow B = \frac{48}{34} = \frac{24}{17}$

$7C + D = 0 \rightarrow D = -7C$

$-C - 49C = 50 \rightarrow -50C = 50 \rightarrow C = -1 \rightarrow D = 7$

$5E = -25 \rightarrow E = -5$

$-20 + 5F = 0 \rightarrow F = 4$

$70 + 8 + 5G = 0 \rightarrow G = -\frac{78}{5}$

(3) (3) (3)

$$y = \left(\tfrac{1}{2}t^3 + \tfrac{24}{17}t^2\right)e^{-t} - \cos\tfrac{1}{2}t + 7\sin\tfrac{1}{2}t - 5t^2 + 4t - \tfrac{78}{5}$$

$$+ c_1e^{-t} + c_2te^{-t} + c_3e^{t}\cos\tfrac{1}{2}t + c_4e^{t}\sin\tfrac{1}{2}t$$ (1½)